GRADED MORITA EQUIVALENCES FOR FROBENIUS KOSZUL ALGEBRAS AND SYMMETRIC ALGEBRAS (Cohomology theory of finite groups and related topics)

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ABSTRACT. Let k be an algebraically closed field of characteristic 0 and Λ a finite-dimensional k-algebra. In this report, we show that, for a Frobenius Koszul algebra Λ with $(\operatorname{rad}\Lambda)^4 = 0$, there exists a symmetric Koszul algebra S such that Λ and S are graded Morita equivalent. This result tells us a new difference between the category of modules over non-graded algebras and the category of modules over graded algebras. This proof is given by the methods of non-commutative algebraic geometry throughout a Koszul duality.

1. AS-regular algebras and geometric algebras

Through this report, let k be an algebraically closed field of characteristic 0 and A a connected graded k-algebra finitely generated in degree 1. That is, $A = T(V)/I$, where V is a k-vector space, $T(V)$ is the tensor algebra of V and I is a two-sided ideal of $T(V)$.

In noncommutative algebraic geometry, Artin and Schelter [AS] defined certain regular algebras.

Definition 1.1 ([AS])**.** A connected graded k-algebra A is called a *d-dimensional Artin-Schelter regular* (simply *AS-regular*) *algebra* if A satisfies the following conditions:

(i) $\operatorname{gldim} A = d < \infty$,

(ii) $\operatorname{GKdim} A := \inf\{\alpha \in \mathbb{R} \mid \dim_k(\sum_{i=0}^{n} A_i) \leq n^{\alpha} \text{ for all } n \gg 0\} < \infty$, called the *Gelfand-Kirillov dimension* of A,

(iii) (*Gorenstein condition*) $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & (i = d), \\ 0 & (i \neq d). \end{cases}$

For example, if A is a graded commutative algebra, A is an n-dimensional AS-regular algebra if and only if A is isomorphic to a polynomial ring $k[x_1, \dots, x_n]$. For example, let A is a grade k-algebra

$$k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx) \quad (\alpha\beta\gamma \neq 0, 1).$$

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Then, A is a 3-dimensional quadratic AS-regular algebra ([M, Example 4.10], [IM1, Theorem 3.1]).

Remark 1.2. It is known from [AS] that

- For any 1-dimensional AS-regular algebra A, A is isomorphic to $k[x]$ as graded k-algebras.
- For any 2-dimensional AS-regular algebra A,
$$A \cong k[x,y] := k\langle x,y\rangle/(-x^2+xy-yx),$$
or
$$A \cong k_\lambda[x,y] := k\langle x,y\rangle/(xy-\lambda yx) \quad (\lambda \in k \setminus \{0\}),$$
where $k_\lambda[x,y] \cong k_{\lambda'}[x,y]$ if and only if $\lambda' = \lambda^{\pm 1}$.

Any 3-dimensional AS-regular algebra A is graded algebra isomorphic to an algebra of the form

$k\langle x,y,z\rangle/(f_1,f_2,f_3)$ (quadratic case) , or $k\langle x,y\rangle/(g_1,g_2)$ (cubic case)

where f_i are homogeneous polynomials of degree 2 and g_i are homogeneous polynomials of degree 3 ([AS, Theorem 1.5 (i)]). In this report, we are interested in a quadratic case.

Let A be a graded k-algebra. A graded A-module M has a *linear resolution* if a free resolution of M is as follows:

$$\cdots \longrightarrow \bigoplus A(-2) \longrightarrow \bigoplus A(-1) \longrightarrow \bigoplus A \longrightarrow M \longrightarrow 0.$$

A graded k-algebra A is called *Koszul* when k has a linear resolution.

Remark 1.3. If A is a Koszul algebra, then $A = T(V)/(R)$ is quadratic, where $R \subset V \otimes_k V$. Moreover, the Ext algebra (the Yoneda algebra) of A $\mathrm{Ext}^*_A(k,k) \cong A^! := T(V^*)/(R^\perp)$ is Koszul, and $A^!$ is called the *Koszul dual* of A, where V^* is the dual space of a finite-dimensional k-vector space V, and $R^\perp := \{f \in V^* \otimes_k V^* \mid f(R) = 0\}$.

Example 1.4. Let A be a graded k-algebra
$$k\langle x,y,z\rangle/(yz-\alpha zy, zx-\beta xz, xy-\gamma yx).$$
Then, the Koszul dual $A^!$ of A is
$$k\langle x,y,z\rangle/(x^2,y^2,z^2,\alpha yz+zy,\beta zx+xz,\gamma xy+yx) \quad (\alpha,\beta,\gamma \in k\setminus\{0\}).$$

For Koszul algebras, by using Koszul duality, Smith [S] proved a relationship between AS-regular Koszul algebras and Frobenius Koszul algebras.

Theorem 1.5 ([S, Proposition 5.10])**.** *Let A be a connected graded Koszul k-algebra. Then A is Koszul AS-regular if and only if the Koszul dual $A^!$ is Frobenius and the complexity of k is finite.*

We remark that, in Theorem 1.5, for a d-dimensional AS-regular Koszul algebra A and the Frobenius Koszul algebra $A^!$, $\operatorname{gldim} A \leq d$ and $\operatorname{GK}\dim A < \infty$ correspond to $(\operatorname{rad} A^!)^3 \neq 0$, $(\operatorname{rad} A^!)^{d+1} = 0$ and $\operatorname{cx}(k) < \infty$, respectively.

Example 1.6. Let

$$A = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx) \quad (\alpha\beta\gamma \neq 0, 1)$$

Then A a 3-dimensional Koszul AS-regular algebra ([M, Example 4.10], [IM1, Theorem 3.1]). Moreover, by Theorem 1.5,

$$A^! = k\langle x, y, z\rangle/(x^2, y^2, z^2, \alpha yz + zy, \beta zx + xz, \gamma xy + yx)$$

is a Frobenius Koszul algebra such that $(\operatorname{rad} A^!)^3 \neq 0$, $(\operatorname{rad} A^!)^4 = 0$ and $\operatorname{cx}(A^!/\operatorname{rad} A^!) = \operatorname{cx}(k) < \infty$.

Now, we consider a homogeneous ideal I of $k\langle x_1, \dots, x_n\rangle$ generated by degree 2 homogeneous polynomials, that is, we treat a quadratic algebra. When a graded k-algebra $A = k\langle x_1, \dots, x_n\rangle/I$ is quadratic, we set

$$\Gamma_A := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0 \text{ for all } f \in I_2\}.$$

Mori [M] introduced a geometric algebra over k as follows.

Definition 1.7 ([M])**.** Let $A = k\langle x_1, \dots, x_n\rangle/I$ be a quadratic k-algebra.

(i) A satisfies (G1) if there exists a pair (E, σ) where E is a closed k-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \operatorname{Aut} E$ such that

$$\Gamma_A = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}.$$

In this case, we write $\mathcal{P}(A) = (E, \sigma)$ called *the geometric pair of* A.

(ii) A satisfies (G2) if there exists a pair (E, σ) where E is a closed k-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \operatorname{Aut} E$ such that

$$I_2 = \{f \in k\langle x_1, \dots, x_n\rangle_2 \mid f(p, \sigma(p)) = 0, \text{ for all } p \in E\}.$$

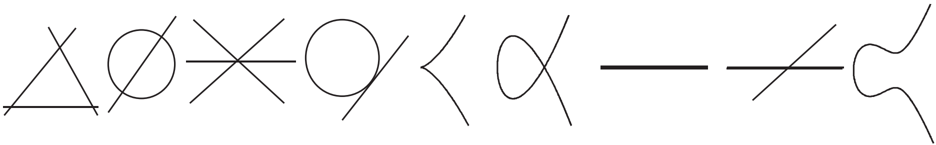
In this case, we write $A = \mathcal{A}(E, \sigma)$.

(iii) A is called *geometric* if A satisfies both (G1) and (G2), and $A = \mathcal{A}(\mathcal{P}(A))$.

Note that, if A satisfies (G1), A determines the pair (E, σ) by using Γ_A. Conversely, if A satisfies (G2), A is determined by the pair (E, σ).

Artin-Tate-Van den Bergh [ATV] found a nice one-to-one correspondence between the set of 3-dimensional AS-regular algebras A and the set of regular geometric pairs (E, σ) where E is a scheme and $\sigma \in \operatorname{Aut}_k E$, so the classification of 3-dimensional AS-regular algebras reduces to the classification of geometric pairs. In paticular, we consider 3-dimensional quadratic AS-regular algebras.

Theorem 1.8 ([ATV]). *Every 3-dimensional quadratic AS-regular algebra A is geometric. Moreover, when $\mathcal{P}(A) = (E, \sigma)$, the point scheme E is either the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$ as follows:*



Artin-Tate-Van den Bergh [ATV] gave a partial list of regular geometric triples. In [IM1], we give all possible defining relations of 3-dimensional quadratic AS-regular algebras. Moreover, we classify them up to isomorphism and up to graded Morita equivalence in terms of their defining relations in the case that their point schemes are not elliptic curves ([IM1, Theorems 3.1, 3.2]). In the case that their point schemes are elliptic curves, we give conditions for isomorphism and graded Morita equivalence in terms of geometric data ([IM1, Theorems 4.9, 4.20]).

2. Calabi-Yau algberas and superpotentials

Here, the definition of Calabi-Yau algebras is as follows:

Definition 2.1 ([G]). Let Λ be a k-algebra. Λ is called *d-dimensional Calabi-Yau* if Λ satisfies the following conditions:

(i) $\mathrm{pd}_{\Lambda^{\mathrm{e}}}\Lambda = d < \infty$,

(ii) $\mathrm{Ext}^i_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}}) = \begin{cases} \Lambda & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$

where $\Lambda^{\mathrm{e}} = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ is the enveloping algebra of Λ.

Now, we recall the definitions of superpotentials and derivation-quotient algebras from [BSW] and [MS]. For simplicity, we consider the case for $n = 3$. For a finite-dimensional k-vector space V, we define the k-linear map φ: $V^{\otimes 3} \longrightarrow V^{\otimes 3}$ by

$$\phi(v_1 \otimes v_2 \otimes v_3) := v_3 \otimes v_1 \otimes v_2.$$

Definition 2.2 ([BSW], [MS]). If $\phi(w) = w$ for $w \in V^{\otimes 3}$, then w is called *superpotential*. Moreover, we set k-vector space $V = \langle x_1, x_2, x_3 \rangle$. For $w \in V^{\otimes 3}$, there exists $w_i \in V^{\otimes 2}$ such that $w = \sum_{i=1}^{3} x_i \otimes w_i$. Then, we define $\partial_{x_i} w := w_i$, and,

$$\mathcal{D}(w) := T(V)/(\partial_{x_i} w)\ (i = 1, 2, 3)$$

is called the *derivation-quotient algebra* of w, where $T(V)$ is the tensor algebra of V.

Theorem 2.3 ([RRZ]). *Let A be a quadratic algebra. Then A is a Calabi-Yau algebra if and only if the Nakayama automorphism ν of $A^!$ is identity, that is, $A^!$ is symmetric.*

Definition 2.4 ([MS], [IM2, Definition 2.8]). For a superpotential $w \in V^{\otimes 3}$ and $\tau \in \mathrm{GL}(V)$,

$$w^\tau := (\tau^2 \otimes \tau \otimes \mathrm{id})(w)$$

is called a *Mori-Smith twist* (*MS twist*) of w by τ.

For a potential $w \in V^{\otimes 3}$, we set

$$\mathrm{Aut}\,(w) := \{\tau \in \mathrm{GL}(V) \mid (\tau^{\otimes 3})(w) = \lambda w,\, \exists \lambda \in k \setminus \{0\}\}.$$

Then it follows that $\mathrm{Aut}\,(w) = \mathrm{Aut}\,\mathcal{D}(w)$.

Proposition 2.5 ([MS, Proposition 5.2]). *For a superpotential $w \in V^{\otimes 3}$ and $\tau \in \mathrm{Aut}\,(w)$, we have that $\mathcal{D}(w^\tau) \cong \mathcal{D}(w)^\tau$.*

Example 2.6. For $w \in V^{\otimes 3}$, we consider

$$w = (xyz + yzx + zxy) - \lambda(zyx + yxz + xzy) \quad (\lambda \in k\backslash\{0\}).$$

Then, we see that

$$\begin{aligned}\phi(w) &= \phi((xyz + yzx + zxy) - \lambda(zyx + yxz + xzy)) \\ &= (zxy + xyz + yzx) - \lambda(xzy + zyx + yxz) \\ &= (xyz + yzx + zxy) - \lambda(zyx + yxz + xzy) = w.\end{aligned}$$

So, w is a superpotential. Also,

$$\partial_x w = yz - \lambda zy,\ \partial_y w = zx - \lambda xz,\ \partial_z w = xy - \lambda yx.$$

Therefore, the derivation-quotient algebra $\mathcal{D}(w)$ is as follows:

$$\begin{aligned}\mathcal{D}(w) &= k\langle x, y, z\rangle/(\partial_x w, \partial_y w, \partial_z w) \\ &= k\langle x, y, z\rangle/(yz - \lambda zy, zx - \lambda xz, xy - \lambda yx).\end{aligned}$$

Moreover, taking $\tau := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \mathrm{GL}_3(V)$. Calculating the MS twist of w by τ,

$$\begin{aligned}w^\tau &= (\tau^2 \otimes \tau \otimes \mathrm{id})(w) \\ &= (\alpha^2\beta xyz + \beta^2\gamma yzx + \alpha\gamma^2 zxy) - \lambda(\beta\gamma^2 zyx + \alpha\beta^2 yxz + \alpha^2\gamma xzy).\end{aligned}$$

3. Main result and example

In this section, we describe our main result and one example.

Theorem 3.1. [IM2, Theorem 4.4] *For every* 3*-dimensional quadratic AS-regular algebra A, there exists a Calabi-Yau AS-regular algebra C such that A and C are graded Morita equivalent.*

The following corollary is our main result in this report. By using Remark 1.2, the fatc that $k[x]$ is a Calabi-Yau algebra, Theorem 1.5 and Theorem 3.1, the following result immediately holds.

Corollary 3.2. *For a Frobenius Koszul algebra Λ with $(\operatorname{rad}\Lambda)^4 = 0$, there exists a symmetric Koszul algebra S such that Λ and S are graded Morita equivalent.*

Remark 3.3. This result tells us a new difference between the category of modules over non-graded algebras and the category of modules over graded algebras.

Example 3.4. Suppose that (E, σ) is a geometric pair where E is a union of three lines making a triangle in $\mathbb{P}^2$ and $\sigma \in \operatorname{Aut} E$ stabilizes each component. That is, $E = \mathcal{V}(xyz)$ and

$$\begin{cases} \sigma(\mathcal{V}(x)) = \mathcal{V}(x), \\ \sigma(\mathcal{V}(y)) = \mathcal{V}(y), \\ \sigma(\mathcal{V}(z)) = \mathcal{V}(z). \end{cases}$$

Considering $A = \mathcal{A}(E, \sigma)$ corresponding to E and $\sigma \in \operatorname{Aut} E$,

$$A = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$$

is 3-dimensional quadratic AS-regular ($\alpha\beta\gamma \neq 0, 1$) (see Example 1.6).

For $\lambda := \sqrt[3]{\alpha\beta\gamma} \in k\backslash\{0\}$, we take a superpotential w as

$$w = (xyz + yzx + zxy) - \lambda(zyx + yxz + xzy).$$

Also, we take $\tau := \begin{pmatrix} \sqrt[3]{\beta\gamma^{-1}} & 0 & 0 \\ 0 & \sqrt[3]{\gamma\alpha^{-1}} & 0 \\ 0 & 0 & \sqrt[3]{\alpha\beta^{-1}} \end{pmatrix} \in \mathrm{GL}\,(3, k)$. Then, the MS twist w^τ by τ is as follows:

$$\begin{aligned} w^\tau &= (\tau^2 \otimes \tau \otimes \mathrm{id})(w) \\ &= \sqrt[3]{\alpha^{-1}\beta^2\gamma^{-1}}xyz + \sqrt[3]{\alpha^{-1}\beta^{-1}\gamma^2}yzx + \sqrt[3]{\alpha^2\beta^{-1}\gamma^{-1}}zxy \\ &\quad - \sqrt[3]{\alpha^2\beta^{-1}\gamma^2}zyx - \sqrt[3]{\alpha^{-1}\beta^2\gamma^2}yxz - \sqrt[3]{\alpha^2\beta^2\gamma^{-1}}xzy. \end{aligned}$$

Therefore, the derivation-quotient algebra $\mathcal{D}(w^\tau)$

$$\mathcal{D}(w^\tau) = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$$

and we have a graded k-algebra isomorphism $A \cong \mathcal{D}(w^\tau)$. By calculation, we see $w \in \mathrm{Aut}(w)$. So, by Proposition 2.5, $\mathcal{D}(w^\tau)$ is isomorphic to $\mathcal{D}(w)^\tau$ as graded algebras. Also, by [Z, Theorem 3.5], $\mathcal{D}(w)^\tau$ is equivalent to $\mathcal{D}(w)$. Since the AS regularity is invariant under Zhang's twist by [Z, Theorem 5.11 (b)], A is equivalent to the Calabi-Yau AS-regular algebra

$$C := \mathcal{D}(w) = k\langle x, y, z\rangle/(yz - \lambda zy, zx - \lambda xz, xy - \lambda yx)$$

as graded Morita equivalent. Note that, by calculation, we have the Kozul dual $\Lambda := A^!$ of A is

$$k\langle x, y, z\rangle/(x^2, y^2, z^2, zy + \alpha xy, xz + \beta zx, yx + \gamma xy),$$

and the Koszul dual $S := C^!$ of C is

$$k\langle x, y, z\rangle/(x^2, y^2, z^2, zy + \lambda xy, xz + \lambda zx, yx + \lambda xy).$$

Also, by calculation, we have that the Nakayama automorphism ν_S of S is identity. So, S is a symmetric algebra by Theorem 2.3. Moreover, by using Theorem 1.5, Λ is a Frobenius Koszul algebra with $(\mathrm{rad}\, A^!)^3 \neq 0$, $(\mathrm{rad}\, A^!)^4 = 0$ and $\mathrm{cx}(A^!/\mathrm{rad}\, A^!) = \mathrm{cx}(k) < \infty$. Therefore, the statement of Corollary 3.2 holds.

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